Accuracy and complexity evaluation of defuzzification strategies for the discretised interval type-2 fuzzy set ☆

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Abstract

The work reported in this paper addresses the challenge of the efficient and accurate defuzzification of discretised interval type-2 fuzzy sets. The exhaustive method of defuzzification for type-2 fuzzy sets is extremely slow, owing to its enormous computational complexity. Several approximate methods have been devised in response to this bottleneck. In this paper we survey four alternative strategies for defuzzifying an *interval* type-2 fuzzy set: 1. The Karnik-Mendel Iterative Procedure, 2. the Wu-Mendel Approximation, 3. the Greenfield-Chiclana Collapsing Defuzzifier, and 4. the Nie-Tan Method.

We evaluated the different methods experimentally for accuracy, by means of a comparative study using six representative test sets with varied characteristics, using the exhaustive method as the standard. A preliminary ranking of the methods was achieved using a multi-criteria decision making methodology based on the assignment of weights according to performance. The ranking produced, in order of decreasing accuracy, is 1. the Collapsing Defuzzifier, 2. the Nie-Tan Method, 3. the Karnik-Mendel Iterative Procedure, and 4. the Wu-Mendel Approximation.

Following that, a more rigorous analysis was undertaken by means of the Wilcoxon Nonparametric Test, in order to validate the preliminary test conclusions. It was found that there was no evidence of a significant difference between the accuracy of the Collapsing and Nie-Tan Methods, and between that of the Karnik-Mendel Iterative Procedure and the Wu-Mendel Approximation. However, there was evidence to suggest that the collapsing and Nie-Tan Methods are more accurate than the Karnik-Mendel Iterative Procedure and the Wu-Mendel Approximation.

In relation to efficiency, each method's computational complexity was analysed, resulting in a ranking (from least computationally complex to most computationally complex) as follows: 1. the Nie-Tan Method, 2. the Karnik-Mendel Iterative Procedure (lowest complexity possible), 3. the Greenfield-Chiclana Collapsing Defuzzifier, 4. the Karnik-Mendel Iterative Procedure (highest complexity possible), and 5. the Wu-Mendel Approximation.

Keywords: Interval Type-2 Fuzzy Set, Defuzzification, Type-Reduction, Karnik-Mendel Iterative Procedure, Nie-Tan Method, Greenfield-Chiclana Collapsing Defuzzifier,

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1. Introduction

Interval type-2 fuzzy sets have increasingly been used in applications [2, 3, 9, 10, 19, 28, 32, 40, 45, 46, 50, 58] as they offer a more sophisticated model of uncertainty than their type-1 counterparts [34], whilst lacking the computational complexity of the generalised type-2 fuzzy set [24, 39]. Since the turn of the millennium algorithms based on the Karnik-Mendel Iterative Procedure (KMIP) [38] have become the established interval defuzzification techniques [4, 22, 23, 25, 27, 33, 47].

However the KMIP and associated algorithms are not the only available methods for defuzzification of interval type-2 fuzzy sets. Three alternative methods are

- the Wu-Mendel Approximation,
- the Greenfield-Chiclana Collapsing Defuzzifier, and
- the Nie-Tan Method.

It is timely for the research community to compare the alternatives to determine which are more accurate and efficient, so that practitioners can have evidence to support their choices in practical applications. Starczewski [48] has contrasted the KMIP and the Wu-Mendel Approximation in the context of a Fuzzy Inferencing System (FIS). In this paper we report on experiments using test sets that compare the four aforementioned methods for accuracy. Efficiency is compared by means of analyses of computational complexity. In the experiments the exhaustive method was employed as the standard for accuracy. Two comparative studies were carried out using six representative test sets which varied in their characteristics. A preliminary comparison was made for accuracy using a multi-criteria decision making methodology based on the assignment of weights according to performance, by which the methods were ranked. Following that, a more rigorous analysis was undertaken with respect to accuracy by means of the Wilcoxon Nonparametric Test, in order to validate the preliminary test conclusions.

1.1. Type-2 Fuzzy Set: Definitions

Let X be a universe of discourse. A type-1 fuzzy set A on X is characterised by a membership function $\mu_A : X \to [0,1]$ and can be expressed as follows [54]:

$$A = \{(x, \mu_A(x)) | \, \mu_A(x) \in [0,1] \, \forall x \in X\}. \tag{1}$$

Note that the membership grades of A are crisp numbers. In the following we will use the notation $U = [0,1]$.

Let $\tilde{P}(U)$ be the set of fuzzy sets in U. A type-2 fuzzy set $\tilde{A}$ in X is a fuzzy set whose membership grades are themselves fuzzy [55–57] (Figure 1). This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in U for all x, i.e. $\mu_{\tilde{A}} : X \to \tilde{P}(U)$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | \, \mu_{\tilde{A}}(x) \in \tilde{P}(U) \, \forall x \in X\}. \tag{2}$$

It follows that $\forall x \in X \, \exists J_x \subseteq U$ such that $\mu_{\tilde{A}}(x) : J_x \to U$. Applying (1), we obtain:

$$\mu_{\tilde{A}}(x) = \{(u, \mu_{\tilde{A}}(x)(u)) | \, \mu_{\tilde{A}}(x)(u) \in U \, \forall u \in J_x \subseteq U\}. \tag{3}$$

X is called the primary domain and J_x the primary membership of x while U is known as the secondary domain and $\mu_{\tilde{A}}(x)$ the secondary membership of x.

Putting (2) and (3) together we obtain

$$\tilde{A} = \{(x,(u,\mu_{\tilde{A}}(x)(u)))|\ \mu_{\tilde{A}}(x)(u) \in U, \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \tag{4}$$

This *vertical representation* of a type-2 fuzzy set is used to define the concept of an *embedded set* of a type-2 fuzzy set (Definition 7), which is fundamental to the definition of the *centroid* of a type-2 fuzzy set (Definition 8). Alternative notations may be found in [1].

Definition 1 (Interval Type-2 Fuzzy Set)**.** *An* interval type-2 fuzzy set *is a type-2 fuzzy set whose secondary membership grades are all* 1.

In the interval case, Equation 4 reduces to:

$$\tilde{A} = \{(x,(u,1)),\ \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \tag{5}$$

With no loss of generality it is assumed that the type-2 fuzzy set is contained within a unit cube and may be viewed as a surface represented by (x,u,z) co-ordinates.

Definition 2 (Footprint Of Uncertainty)**.** *FOU stands for* Footprint Of Uncertainty (FOU)*, the projection of the type-2 fuzzy set onto the $x-u$ plane.*

Definition 3 (Lower Membership Function)**.** *The* Lower Membership Function (LMF) *of a type-2 fuzzy set is the type-1 membership function associated with the lower bound of the FOU.*

Definition 4 (Upper Membership Function)**.** *The* Upper Membership Function (UMF) *of a type-2 fuzzy set is the type-1 membership function associated with the upper bound of the FOU.*

Definition 5 (Vertical Slice)**.** *A* vertical slice *is a plane which intersects the x-axis (primary domain) and is parallel to the u-axis (secondary domain).*

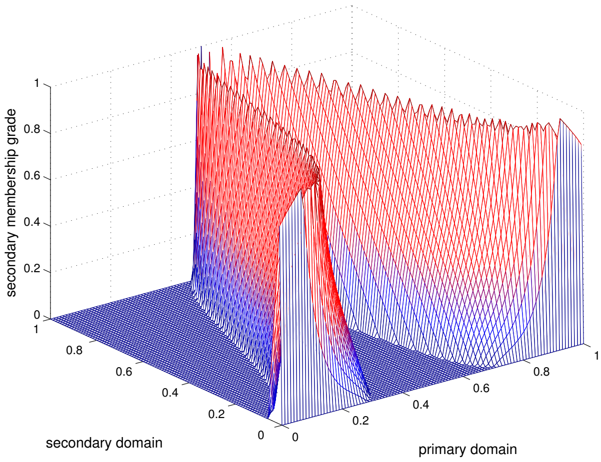
Definition 6 (Degree of Discretisation)**.** *The* degree of discretisation *is the separation of the slices.*

1.2. Mamdani Fuzzy Inferencing Systems

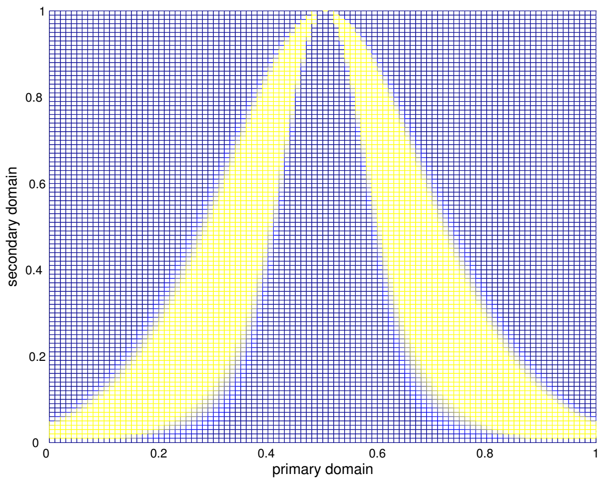
In the Mamdani Fuzzy Inferencing System, a crisp numerical input passes through three stages: fuzzification, inferencing, and finally defuzzification. The output of inferencing is a fuzzy set known as the *aggregated set*. During the defuzzification stage the aggregated set is converted into a crisp number, the final result of the processing of the FIS. Figure 2 provides a representation of a Mamdani-style *type-2* FIS, showing the defuzzification stage as consisting of two parts, *type-reduction* and defuzzification proper. Type-reduction is the procedure by which a type-2 fuzzy set is converted to a type-1 fuzzy set known as the *Type-Reduced Set (TRS)*. This set is then defuzzified to give a crisp number. The additional stage of type-reduction distinguishes the type-2 FIS from its type-1 counterpart and has been a processing bottleneck in type-2 fuzzy inferencing [7, 13, 17, 24] because it relies on finding the centroids of an extraordinarily large number of type-1 fuzzy sets (embedded sets) into which the type-2 fuzzy set is decomposed.

1.3. Structure of the Paper

In the next section exhaustive defuzzification is described. Following that, in Section 3, the various alternatives to exhaustive defuzzification are presented. Section 4 concerns the experimental evaluation of the different methods for accuracy. Section 5 is a statistical comparison of the test results in relation to accuracy, and Section 6 presents computational complexity analyses of the methods. In Section 7 conclusions are drawn and suggestions made for further work.



(a) 3-D representation.



(b) FOU.

Figure 1: Type-2 fuzzy set: Gaussian primary membership function, triangular secondary membership functions; degree of discretisation of primary and secondary domains is 0.01; defuzzified value = 0.5. In Fig. 1(b) the FOU is clearly depicted, bounded below by the LMF and above by the UMF.

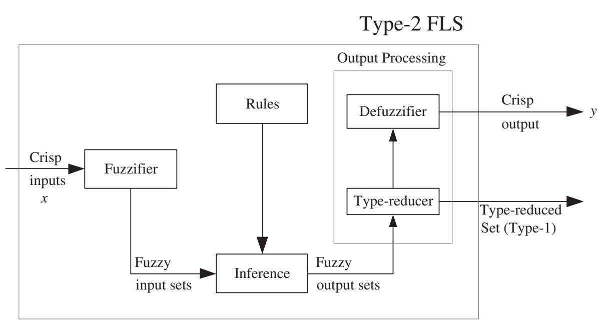


Figure 2: Type-2 FIS (from Mendel [38]).

2. Exhaustive Defuzzification

For type-1 fuzzy sets defuzzification is a straightforward matter. There are several defuzzification techniques available, including the centroid, the centre of maxima and the mean of maxima [31]. Type-2 defuzzification normally consists of two stages [38]:

1. Type-reduction, which converts a type-2 fuzzy set to a type-1 fuzzy set, and
2. defuzzification of the type-1 fuzzy set.

Mathematically, the type-reduction algorithm depends upon the *Extension Principle* [55], which generalises operations defined for crisp numbers to type-1 fuzzy sets. Type-2 defuzzification techniques therefore derive from and incorporate type-1 defuzzification methods[1]. The research presented in this paper makes use solely of the centroid method of type-1 defuzzification [29].

2.1. The Wavy-Slice Representation Theorem

The concept of an *embedded type-2 set* or *wavy-slice* [39] is crucial to type-reduction. An embedded type-2 fuzzy set (or 'embedded set' for short) is a special kind of type-2 fuzzy set. It relates to the type-2 fuzzy set in which it is embedded in this way: For every primary domain value, x, there is a unique secondary domain value, u, plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)$.

Example 1. *In Figure 3 we have identified two embedded sets of a type-2 fuzzy set with primary and secondary domain degree of discretisation of* 0.1*. The embedded set* $\tilde{P}$ *is represented by pentagonal, pointed flags, and embedded set* $\tilde{Q}$ *is symbolised by quadrilateral shaped flags.*

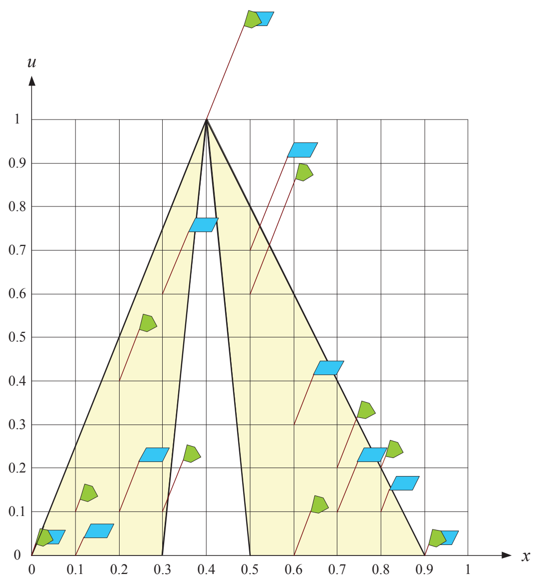


Figure 3: Two embedded sets, indicated by different flag styles. The flag height reflects the secondary membership grade. The degree of discretisation of the primary and secondary domains is 0.1. The shaded region is the FOU.

[1]Geometric defuzzification [8] is exceptional among type-2 defuzzification methods in not involving type-reduction and therefore not requiring type-1 defuzzification.

Definition 7 (Embedded Set)**.** *Let $\tilde{A}$ be a type-2 fuzzy set in X. For discrete universes of discourse X and U,* an embedded type-2 set *$\tilde{A}_e$ of $\tilde{A}$ is defined as the following type-2 fuzzy set*

$$\tilde{A}_e = \{(x_i, (u_i, \mu_{\tilde{A}}(x_i)(u_i)))|\ \forall i \in \{1,\ldots,N\} :\ x_i \in X\ \ u_i \in J_{x_i} \subseteq U\}. \tag{6}$$

$\tilde{A}_e$ contains exactly one element from J_{x_1}, J_{x_2}, $\ldots$, J_{x_N}, namely u_1, u_2, $\ldots$, u_N, each with its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_1)$, $\mu_{\tilde{A}}(x_2)(u_2)$, $\ldots$, $\mu_{\tilde{A}}(x_N)(u_N)$.

Mendel and John have shown that a type-2 fuzzy set can be represented as the union of its type-2 embedded sets [39, page 121]. This powerful result is known as the type-2 fuzzy set '*Representation Theorem*' or '*Wavy-Slice Representation Theorem*'; in [39] it was derived without reference to the Extension Principle [55]. Bringing a conceptual simplicity to the manipulation of type-2 fuzzy sets, it is applied to give simpler derivations of results previously obtained through the Extension Principle.

Theorem 1 (Representation Theorem [39])**.** *Let $\tilde{A}_e^j$ denote the jth type-2 embedded set for type-2 fuzzy set $\tilde{A}$, i.e.,*

$$\tilde{A}_e^j \equiv \left\{ \left(x_i, \left(u_i^j, \mu_{\tilde{A}}(x_i)(u_i^j)\right)\right), i = 1,\ldots,N\right\} \tag{7}$$

where $\{u_i^j,\ldots,u_N^j\} \in J_{x_i}$. Then $\tilde{A}$ can be represented as the union of its type-2 embedded sets, i.e.,

$$\tilde{A} = \sum_{j=1}^{n} \tilde{A}_e^j \tag{8}$$

where

$$n \equiv \prod_{i=1}^{N} M_i, \tag{9}$$

and M_i is the number of values into which the i^{th} vertical slice has been discretised.

For a generalised type-2 fuzzy set the first stage of type-2 defuzzification is to create the *Type-Reduced Set (TRS)*. Assuming that the primary domain X has been discretised, the TRS of a type-2 fuzzy set is defined as follows:

Definition 8. *The TRS associated with a type-2 fuzzy set $\tilde{A}$ with primary domain X discretised into N points $X = \{x_1, x_2, \ldots, x_N\}$, is*

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{N} x_i \cdot u_{k_i}}{\sum_{i=1}^{N} u_{k_i}}, \mu_{\tilde{A}}(x_1)(u_{k_1}) * \ldots * \mu_{\tilde{A}}(x_N)(u_{k_N}) \right) \middle| \forall (u_{k_1}, u_{k_2}, \ldots, u_{k_N}) \right.$$
$$\left. \in J_{x_1} \times J_{x_2} \times \ldots \times J_{x_N} \subseteq U^N \right\}. \tag{10}$$

In order for this definition of the TRS to be meaningful, the domain X must be numeric in nature. The TRS is a type-1 fuzzy set in U and its computation in practice requires the secondary domain U to be discretised as well. Algorithm 1 (adapted from Mendel [38]) is used to compute the TRS of a type-2 fuzzy set.

2.2. *Exhaustive Type-Reduction*

Mendel and John's Representation Theorem (Subsection 2.1) provides a precise, straightforward method for type-2 defuzzification. Though Definition 8 does not explicitly mention embedded sets, they appear implicitly. When this definition is presented in algorithmic form (Algorithm 1), explicit mention is made of embedded sets. As *every* embedded set is processed, this stratagem has become known as the *exhaustive method* [15]. Discretisation inevitably brings with it an element of approximation. However the exhaustive method does not introduce further inaccuracies subsequent to discretisation.

Exhaustive type-reduction processes every embedded set in turn. Each embedded set is defuzzified as a type-1 fuzzy set. The defuzzified value is paired with the minimum secondary membership grade of the embedded set; the set of ordered pairs constitutes the TRS. The major shortcoming of this method is its computational complexity.

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set (the TRS)

forall the *embedded sets* **do**
- calculate the primary domain value (x) of the type-1 centroid of the type-2 embedded set ;
- find the minimum secondary membership grade (z) ;
- pair the secondary grade (z) with the primary domain value (x) to give set of ordered pairs (x, z) {some values of x may correspond to more than one value of z} ;

end
forall the *primary domain (x) values* **do**
- select the maximum secondary grade {make each x correspond to a unique value} ;

end

Algorithm 1: Type-reduction of a discretised type-2 fuzzy set to a type-1 fuzzy set, adapted from Mendel [38].

3. Interval Type-Reduction Strategies

As the interval type-2 fuzzy set is a special case of the generalised type-2 fuzzy set, all generalised methods of defuzzification [18], [35], [37] are applicable to interval type-2 fuzzy sets. However, this section concerns techniques specifically developed as interval methods.

For the TRS of an interval type-2 fuzzy set, Definition 8 reduces to:

Definition 9 (TRS of an Interval Type-2 Set). *The TRS associated with an interval type-2 fuzzy set $\tilde{A}$ with primary domain X discretised into N points $X = \{x_1, x_2, \ldots, x_N\}$, is*

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{N} x_i \cdot u_{k_i}}{\sum_{i=1}^{N} u_{k_i}}, 1 \right) \middle| \, \forall (u_{k_1}, u_{k_2}, \ldots, u_{k_N}) \in J_{x_1} \times J_{x_2} \times \ldots \times J_{x_N} \subseteq U^N \right\}. \tag{11}$$

3.1. *The Karnik-Mendel Iterative Procedure*

The most widely adopted method for type-reducing an interval type-2 fuzzy set is the *Karnik-Mendel Iterative Procedure* [26]. The result of type-reduction of an interval type-2 fuzzy set is an interval set (which is a particular case of a type-1 fuzzy set)[2], with the defuzzified value of the type-2 fuzzy set located at the midpoint. The iterative procedure is an efficient method for finding the endpoints of the interval. There is an element of approximation in the defuzzified value, as in general the TRS tuples are not symmetrically distributed over the interval[3].

Since the publication of the KMIP, various enhanced versions have been proposed [51], [36]. They differ somewhat in their search strategy. Wu and Nie [52] present five variations, and go on to compare them experimentally in relation to efficiency, finding the optimum algorithm to be the *Enhanced Iterative Algorithm with Stop Condition (EIASC)* [52, Section III] (Algorithm 2). Wu and Nie's Matlab^{TM} code is to be found in Appendix A of [52].

A triangular generalised type-2 system with a defuzzification algorithm based on the KMIP has been developed by Starczewski [49]. Molaeezadeh et al. [41] have proposed a '2uFunction' representation for generalised type-2 fuzzy sets; the FIS based on this representation uses KMIP defuzzification. Kumbasar et al. [30] have decomposed an interval type-2 FIS into several interval type-2 fuzzy subsystems that employ the KMIP as type-reducer.

3.2. The Wu-Mendel Approximation

In [53] Wu and Mendel provide a closed form formula for the centroid of a type-2 interval fuzzy set by calculating approximations[4] to the endpoints (or uncertainty bounds) of the type-reduced interval. The algorithm [53, Appendix III, page 635] is set out below (Algorithm 3). The parameters of the Wu-Mendel Approximation, as used in the algorithm, are shown diagrammatically in Figure 4.

3.3. The Greenfield-Chiclana Collapsing Defuzzifier

A computationally simple alternative to the exhaustive method is the *Greenfield-Chiclana Collapsing Defuzzifier (GCCD)* [14]. This technique converts an interval type-2 fuzzy set into a type-1 fuzzy set which approximates to the *Representative Embedded Set (RES)*, whose defuzzified value is by definition equal to that of the original type-2 set (Fig. 5). We term this type-1 set the *Representative Embedded Set Approximation (RESA)*. As a type-1 set, the RESA may then be defuzzified straightforwardly. Hence the collapsing process reduces the computational complexity of type-2 defuzzification.

Full details of the collapsing algorithm may be found at [14]. We formally state the Simple[5] Representative Embedded Set Approximation:

Theorem 2 (Simple Rep. Embedded Set Approx.)**.** *The membership function of the* embedded set *R derived by dynamically collapsing slices of a discretised type-2 interval fuzzy set $\tilde{A}$, having lower membership function L, and upper membership function U, is:*

$$\mu_R(x_i) = \mu_L(x_i) + r_i \tag{12}$$

[2] The endpoints of the interval are termed 'uncertainty bounds' as the length of the TRS is regarded as a measure of the uncertainty pertaining to the aggregated set [53, page 622].

[3] As discretisation is made finer the gaps between the tuples decrease, and in the limiting case (degree of discretisation = 0) the tuples form a continuous line. In this case the defuzzified value is located exactly at the midpoint of the interval. However, since the KMIP is a search algorithm, it is not applicable in the continuous case, and therefore it is not guaranteed that the exact centroid will be obtained.

[4] This contrasts with the KMIP, which, in the discretised case, is intended to find the endpoints accurately.

[5] In [14], we used the term 'simple' to describe an interval type-2 fuzzy set in which each vertical slice consists of only two points, corresponding to L and U. The term is redundant in the context of this paper.

Input: a discretised interval type-2 fuzzy set
Output: the endpoints of the TRS

1 set x_i $i = 1, 2, \ldots, N$ to be the domain values of the vertical slices ;
2 set L_i to be the lower membership grade of J_i ;
3 set U_i to be the upper membership grade of J_i ;
4 {to compute the left endpoint} ;
5 initialise $a = \sum_{i=1}^{N} x_i L_i$;
6 initialise $b = \sum_{i=1}^{N} L_i$;
7 initialise $y_l = x_N$ {left endpoint} ;
8 initialise $l = 0$;
9 calculate $l = l + 1$;
10 calculate $a = a + x_l(U_l - L_l)$;
11 calculate $b = b + U_l - L_l$;
12 calculate $c = \frac{a}{b}$;
13 **if** $c > y_l$ **then**
14   set $l = l - 1$;
15   stop ;
16 **end**
17 **otherwise**
18   set $y_l = c$;
19   go to Step 9 ;
20 **endsw**
21 {to compute the right endpoint} ;
22 initialise $a = \sum_{i=1}^{N} x_i L_i$;
23 initialise $b = \sum_{i=1}^{N} L_i$;
24 initialise $y_r = x_1$ {right endpoint} ;
25 initialise $r = N$;
26 calculate $a = a + x_r(U_r - L_r)$;
27 calculate $b = b + U_r - L_r$;
28 calculate $c = \frac{a}{b}$;
29 calculate $r = r - 1$;
30 **if** $c < y_r$ **then**
31   set $r = r + 1$;
32   stop ;
33 **end**
34 **otherwise**
35   set $y_r = c$;
36   go to Step 26 ;
37 **endsw**

Algorithm 2: EIASC.

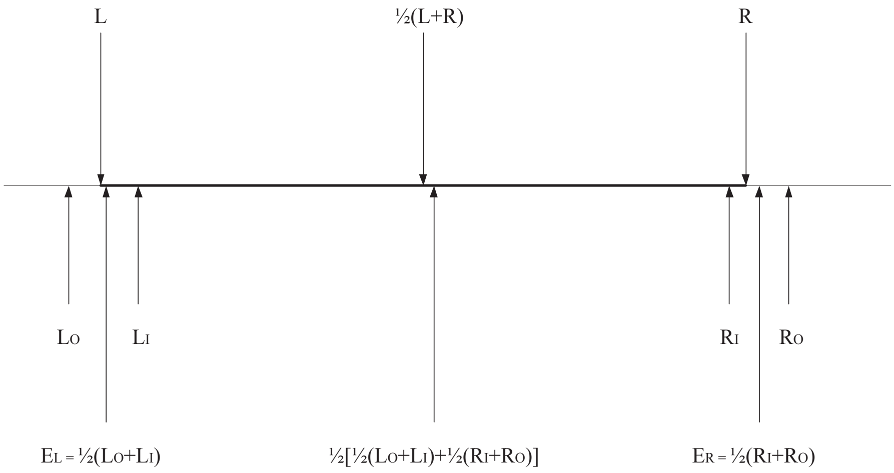


Figure 4: The Wu-Mendel Approximation (adapted from [53]). The KMIP finds the left uncertainty bound L and the right uncertainty bound R. The defuzzified value is taken to be the mean of L and R. The Wu-Mendel Approximation approximates these values to E_L and E_R respectively. E_L is mid way between L_O, the left outer-bound and L_I, the left inner-bound. Similarly E_R is mid way between R_I, the right inner-bound, and R_O, the right outer-bound. As with the KMIP, the defuzzified value is taken to be the mean of E_L and E_R.

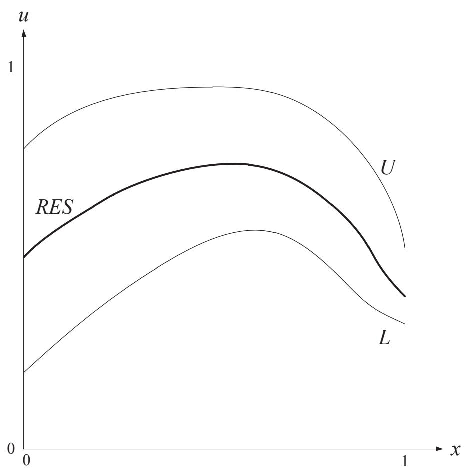


Figure 5: A Representative Embedded Set (continuous case).

Input: a discretised interval type-2 fuzzy set
Output: approximations to the endpoints of the TRS

set x_i $i = 1, 2 \ldots, N$ to be the domain values of the vertical slices ;
set L_i to be the lower membership grade of J_i ;
set U_i to be the upper membership grade of J_i ;
set L_I to be the left inner-bound ;
set R_I to be the right inner-bound ;
set L_O to be the left outer-bound ;
set R_O to be the right outer-bound ;
set E_L to be the left endpoint ;
set E_R to be the right endpoint ;
calculate $l = \dfrac{\sum_i L_i x_i}{\sum_i L_i}$ {defuzzify the lower membership function} ;
calculate $u = \dfrac{\sum_i U_i x_i}{\sum_i U_i}$ {defuzzify the upper membership function} ;
calculate $L_I = min(l, u)$;
calculate $R_I = max(l, u)$;
calculate $L_O = L_I - \dfrac{\sum_i (U_i - L_i)}{\sum_i U_i \cdot \sum_i L_i} \cdot \dfrac{\sum_i L_i x_i \cdot \sum_i U_i (1 - x_i)}{\sum_i L_i x_i + \sum_i U_i (1 - x_i)}$;
calculate $R_O = R_I + \dfrac{\sum_i (U_i - L_i)}{\sum_i U_i \cdot \sum_i L_i} \cdot \dfrac{\sum_i U_i x_i \cdot \sum_i L_i (1 - x_i)}{\sum_i U_i x_i + \sum_i L_i (1 - x_i)}$;
calculate $E_L = \dfrac{L_O + L_I}{2}$;
calculate $E_R = \dfrac{R_O + R_I}{2}$;

Algorithm 3: The Wu-Mendel Approximation.

with

$$r_i = \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}, \tag{13}$$

and $b_i = \mu_U(x_i) - \mu_L(x_i),\ r_0 = 0.$

This is an iterative formula. Collapsing proceeds vertical slice by vertical slice. The first slice is collapsed, the first u-value of the RESA calculated, the next slice is collapsed and the second u-value of the RESA calculated, and so on until all the slices have been collapsed. In this formula b_i is the blur for vertical slice i, i.e. the difference between the upper membership function and the lower membership function for slice i. r_i is the amount by which the u-value of L must be increased to give the u-value of the RESA R.

There are many variants of the collapsing strategy, since slice collapse may proceed in any slice order. The different variants give rise to slightly different defuzzified values [16].

3.4. The Nie-Tan Method

Nie and Tan [43] describe an efficient type-reduction method for interval type-2 fuzzy sets, which involves taking the mean of the lower and upper membership functions of the interval set, so creating a type-1 fuzzy set. Symbolically, $\mu_T(x_i) = \frac{1}{2}(\mu_L(x_i) + \mu_U(x_i))$, where T is the resultant type-1 fuzzy set.

3.5. Summary of Interval Methods

Table 1 summarises the characteristics of the various methods.

Method	*Method by Design?*	*Generalised?*	*Simplification?*	*Precise?*	*Type-Reducer?*	*Type-Red. to TRS?*	*Search Algorithm?*	*Symmetric?*	*Iterative?*	*Stand-Alone?*
Exhaustive	no	yes	no	yes	yes	yes	no	yes	no	yes
KMIP	yes	no	yes	no	yes	yes	yes	yes	yes	yes
Wu-Mendel	yes	no	no	no	yes	yes	no	yes	no	yes
GCCD (1-pass)	yes	no	no	no	yes	no	no	no	yes	yes
Nie-Tan	yes	no	no	no	yes	no	no	yes	no	yes
CORL (2-pass)	yes	no	no	no	yes	no	no	yes	yes	yes
EIASC	yes	no	no	no	yes	yes	yes	yes	yes	yes

Table 1: Comparing and contrasting the major defuzzification methods for discretised type-2 fuzzy sets.

4. Experimental Evaluation of the Accuracy of the Defuzzification Techniques

In the last section several interval defuzzification strategies have been presented. But which should the application developer choose? In the remainder of this section we report on experiments which evaluate the methods by testing them for accuracy. The test runs were performed in isolation from the rest of the FIS, on six specially created interval test sets. The error of a test run was measured by finding the absolute difference between the resultant defuzzified value and the benchmark exhaustive defuzzified value for the test set in question.

The defuzzification methods were coded in Matlab^{TM} and tested on a laptop with an AMD Turion II Neo K645 CPU, a clock speed of 1.6 GHz, and a 4096MB 1333MHz Dual Channel DDR3 SDRAM, running the MS Windows®7 SP1 Home Premium 64 bit operating system.

The GCCD is best thought of as a family of methods as there are a number of variants. It has been demonstrated practically and theoretically that the two-pass Collapsing Outward Right-Left (CORL) is the most accurate variant [16]. Algorithms based on the KMIP form another family (Subsection 3.1). In [52], Wu and Nie have shown that the most efficient version of the KMIP is EIASC. Accordingly, the experiments reported in this section make use of CORL and EIASC.

Interval Test Sets. Six interval type-2 fuzzy sets were prepared: M, N, S, U, W and X (Appendix A). Test sets M and X were taken from [35, pages 2230 – 2233], and the others devised so that the group as a whole exhibited a wide range of features as set out in Table 2.

Feature	M	N	S	U	W	X
Symmetrical	no	yes	no	no	no	no
Extreme (low or high) defuzzified value	no	yes	yes	no	no	no
FOU with narrow section	no	yes	yes	yes	no	yes
FOU with wide section	no	no	no	yes	yes	yes
$[0,1]$ as support	yes	no	no	yes	yes	no
Normal	yes	yes	yes	yes	no	yes
Lower membership function normal	no	yes	yes	yes	no	no
Piecewise linear lower and upper memb. functions	no	yes	no	no	no	yes
Smooth lower and upper membership functions	no	no	yes	yes	yes	no

Table 2: Features of the interval test sets.

Degree of Discretisation. Each test set was discretised into 5, 9, 11, 17 and 21 vertical slices[6]. Each of the six test sets, at each degree of defuzzification, was defuzzified using each of the four methods of defuzzification to be tested, namely CORL, EIASC, the Nie-Tan Method and the Wu-Mendel Approximation. To provide benchmark values for accuracy the test sets were also defuzzified using the interval exhaustive method (Tables B.15, C.17, D.19, E.21, F.23 and G.25 in Appendices B to G).

4.1. *Accuracy of Interval Methods*

A preliminary analysis was performed based on the measured errors as described above. Each method was applied 5 times to each test set, corresponding to each of the degrees of discretisation. For each test set at each degree of discretisation we can rank the method from best to worst according to the magnitude of the error. A multi-criteria decision making model [5, 6, 20, 21, 44] applicable in this situation consists in assigning utility values to each one of the methods reflecting its position in the ranking obtained. The utility values are aggregated to achieve a final global performance score that is used as a choice value to derive their overall performance ranking.

[6]It is convenient to use odd numbers of vertical slices so that there is always a middle slice to use as the starting point for collapsing.

From Tables B.16, C.18, D.20, E.22, F.24 and G.26 (Appendix B to Appendix G), using weighting rules such that a weighting of 4 is assigned to first place, 3 to second place, 2 to third place, and 1 to fourth place, the accuracy rankings of the four interval methods were derived. These are to be found in Tables 3 to 6. Table 7 is a summary of the weighted scores, from which it can be seen that CORL is the most accurate method, the Nie-Tan Method the second most accurate, EIASC the third most accurate, and the Wu-Mendel Approximation the least accurate.

For each test set each method is applied five times, so for the typical test set there will be 5 first places, 5 second places, 5 third places and 5 fourth places. Test set N is unusual. For this set, owing to ties (Appendix C, Table C.18), there are 9 first places, 2 second places, 4 third places and 5 fourth places.

Though these experiments show CORL to be the superior method in relation to accuracy, the technique's performance was not strong for every test set. CORL was ranked first for accuracy 100% of the time for test sets S, U and W. For test set M, CORL was ranked first 80% of the time. But for test sets N and X, CORL was ranked first only 20% of the time. Even worse, for set N, the 'first' was in fact a 'first equal' with EIASC and the Nie-Tan Method. What might explain the uneven performance of CORL? One factor that stands out immediately is that the sets that CORL performed well with are smooth, whereas the ones for which it performed badly are spiky. For test set M, which is mostly smooth, but contains a downward spike in both its lower and upper membership function, CORL performed quite well $\left(\frac{4}{5}\right)$. Reassuringly, even for the spiky test sets, CORL was not the worst performing method in every run. In fact CORL did not come last in any of the comparisons. In most cases where CORL did not perform the best, the Nie-Tan Method was most accurate; in the remainder of cases EIASC was the most accurate. This matter requires more investigation.

The Nie-Tan Method slightly outperformed EIASC for accuracy. This is surprising since the Nie-Tan Method is conceptually very simple, and involves embedded sets neither in its processing nor its derivation. The Wu-Mendel Approximation did not come first on any occasion; its best performance was second, but in most instances it was the worst performing method.

Distance comparison cannot be considered a rigourous technique because values that differ might not be *significantly different* from a statistical point of view. Greater rigour can be achieved through statistical testing as shown in the next section.

5. Statistical Comparison of the Methods in Relation to Accuracy

The hypothesis that we are testing in this section can be stated as follows:

> *The CORL, EIASC, Nie-Tan and Wu-Mendel Methods do not produce significantly different defuzzified values.*

To compare each pair of interval methods we have to analyse two related samples, the defuzzified values obtained by each method's application to the same six test sets referred to above. The usual parametric test to use in these cases is the t-test applied to the difference scores. However, this test requires for its application the assumptions of normality and independent distribution of the difference scores in the population from which the six test sets are drawn[7]. However, on the one hand, we consider these assumptions to be unjustifiable in our context since there is no evidence to support them, i.e. we have no information about the nature

[7] Although we did not apply any specific random sampling method, we consider the set of six test sets to constitute a sample representative of the whole set of interval type-2 fuzzy sets.

Position	M	N	S	U	W	X	Total	Weighting	Weighted Total
First	4	1	5	5	5	1	21	4	84
Second	0	1	0	0	0	2	3	3	9
Third	1	3	0	0	0	2	6	2	12
Fourth	0	0	0	0	0	0	0	1	0
Grand Total of Weighted Totals									**105**

Table 3: Rankings of CORL in relation to accuracy.

Position	M	N	S	U	W	X	Total	Weighting	Weighted Total
First	0	5	0	0	0	1	6	4	24
Second	1	0	1	0	2	1	5	3	15
Third	4	0	2	1	2	3	12	2	24
Fourth	0	0	2	4	1	0	7	1	7
Grand Total of Weighted Totals									**70**

Table 4: Rankings of EIASC in relation to accuracy.

Position	M	N	S	U	W	X	Total	Weighting	Weighted Total
First	1	3	0	0	0	3	7	4	28
Second	4	1	4	2	1	1	13	3	39
Third	0	1	1	2	3	0	7	2	14
Fourth	0	0	0	1	1	1	3	1	3
Grand Total of Weighted Totals									**84**

Table 5: Rankings of the Nie-Tan Method in relation to accuracy.

Position	M	N	S	U	W	X	Total	Weighting	Weighted Total
First	0	0	0	0	0	0	0	4	0
Second	0	0	0	3	2	1	6	3	18
Third	0	0	2	2	0	0	4	2	8
Fourth	5	5	3	0	3	4	20	1	20
Grand Total of Weighted Totals									**46**

Table 6: Rankings of the Wu-Mendel Approximation in relation to accuracy.

Method	Total Weighted Score
CORL	105
Nie-Tan Method	84
EIASC	70
Wu-Mendel Approx.	46

Table 7: Overall performance of the interval test sets in relation to accuracy.

of the population from which the six test sets are drawn nor do we have any knowledge about any of its parameters. Also, by not requiring these stringent assumptions we can, on the other hand, achieve greater generality in our conclusions. Therefore, we conclude that nonparametric tests are most appropriate in our experimental study; we will use the Wilcoxon Matched-Pairs Signed-Ranks Test [42] to be described in the next subsection.

5.1. Wilcoxon Matched-Pairs Signed-Ranks Statistical Test

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from some unknown continuous distribution function F. Let p be a positive real number, $0 < p < 1$, and let $\xi_p(F)$ denote the quantile of order p for the distribution function F, that is, $\xi_p(F)$ is a solution of $F(x) = p$. For $p = 0.5$, $\xi_{0.5}(F)$ is known as the median of F.

A problem of location is set up by testing the null hypothesis $H_0 : \xi_p(F) = \xi_0$ against one of the alternatives $\xi_p(F) > \xi_0$, $\xi_p(F) < \xi_0$ or $\xi_p(F) \neq \xi_0$. The Wilcoxon Signed-Ranks Test provides a statistical hypothesis test which takes into account the magnitude of the difference between the observations and the hypothesized quantile in order to solve the issue of location.

Let $H_0 : \xi_{0.5}(F) = \xi_0$ be the null hypothesis. Consider the differences $D_i = X_i - \xi_0$, $i = 1, 2, \ldots, n$. Under H_0, the expected number of negative differences will be $n/2$ and negative and positive differences of equal absolute magnitude should occur with equal probability. Consider the absolute values $|D_1|, |D_2|, \ldots, |D_n|$ and rank them from 1 to n. Let T_+ be the sum of ranks assigned to those $D_i's$ that are positive and T_- be the sum of ranks assigned to those $D_i's$ that are negative. It follows that

$$T_+ + T_- = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

so T_+ and T_- are linearly related and offer equivalent criteria. A large value of T_+ indicates that most of the larger ranks are assigned to positive $D_i's$. It follows that large values of T_+ support $H_1 : \xi_{0.5}(F) > \xi_0$. A similar analysis applies to the other two alternatives. So, the test rejects $H_0 : \xi_{0.5}(F) = \xi_0$ to accept $H_1 : \xi_{0.5}(F) > \xi_0$ if $T_+ > c_1$, it rejects H_0 to accept $H_1 : \xi_{0.5}(F) < \xi_0$ if $T_- > c_2$ and it rejects H_0 to accept $H_1 : \xi_{0.5}(F) \neq \xi_0$ if $T_+ > c_3$ or $T_- > c_4$ where c_i are the critical region values.

Under H_0, the common distribution of T_+ and T_- is symmetric with mean $E[T_+] = n(n+1)/4$ and variance $\mathrm{var}[T_+] = n(n+1)(2n+1)/24$. For large n, the standardized T_+ has approximately a standard normal distribution.

In the case of matched-paired data $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ obtained from the application of two treatments (in our case – two interval defuzzification methods) to the same set of subjects (in our case – the set of six test sets), in order to test $H_0 : \xi_{0.5}(F_{X_i - Y_i}) = \xi_0$ against one-sided or two-sided alternatives, the Wilcoxon Test is performed exactly as above by taking $D_i = X_i - Y_i - \xi_0$. In our study we want to test whether the application of the different interval defuzzification methods produces significantly different defuzzified values, i.e. we are testing a null hypothesis with a value $\xi_0 = 0$, $H_0 : \xi_{0.5}(F_{X_i - Y_i}) = 0$. We are testing against the alternative hypothesis of method X being more accurate than method Y, so we will use one-tailed testing $H_1 : \xi_{0.5}(F_{X_i - Y_i}) < 0$.

We assume that two measures with test p-value under the null hypothesis lower than or equal to 0.05 (α) will be considered as significantly different; we refer to it as the test being significant and therefore we conclude that the null hypothesis tested is to be rejected. Otherwise, we fail to reject the null hypothesis.

5.2. Experimental Results

We wanted to test whether there is a significant difference in accuracy between the four interval methods. The interval methods may be paired in six ways. The Wilcoxon Signed Rank Test results are presented in Table 8. The results from the test sets discretised into 21 vertical slices were used in the comparison, as these form the closest approximations to the continuous case.

First Method	Second Method	n	T	Conclusion
CORL	EIASC	6	2	reject H_0; CORL more accurate than EIASC
CORL	Nie-Tan	6	6	cannot reject H_0
CORL	Wu-Mendel	6	0	reject H_0; CORL more accurate than Wu-Mendel
EIASC	Nie-Tan	5	0	reject H_0; Nie-Tan more accurate than EIASC
EIASC	Wu-Mendel	6	7	cannot reject H_0
Nie-Tan	Wu-Mendel	6	0	reject H_0; Nie-Tan more accurate than Wu-Mendel

Table 8: Comparing the errors from the four interval defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with 21 vertical slices. $\alpha = 0.05$. The critical value is 2.

Convergence of Collapsing and Nie-Tan Methods. From Table 8 it can be seen that there is no evidence to reject the null hypothesis when comparing the CORL and Nie-Tan Methods, and therefore we can claim that there is no significant difference between their accuracies. This is to be expected since as the degree of discretisation becomes finer, $||L||$ in the collapsing formula (Equation 13) tends to infinity, making the expression $\|L\| + \sum_{j=1}^{j=i-1} r_j$ also tend to infinity. r_i therefore increases, with $\frac{1}{2}$ as its upper bound. Thus in the continuous case the GCCD computes $\mu_R(x_i) = \mu_L(x_i) + \frac{1}{2}(\mu_U(x_i) - \mu_L(x_i)) = \mu_L(x_i) + \frac{1}{2}\mu_U(x_i) - \frac{1}{2}\mu_L(x_i) = \frac{1}{2}(\mu_L(x_i) + \mu_U(x_i))$. Therefore in the continuous case the collapsing and Nie-Tan methods are equivalent.

Uncertainty Bounds Based Methods. Table 8 indicates that there is no significant difference between the accuracy of EIASC and that of the Wu-Mendel Approximation. This is not surprising as the Wu-Mendel Approximation is intended to be an approximation to the KMIP, which is the forerunner of and gives the same results as EIASC.

Relative Accuracies of the Methods. Table 8 shows that there exists evidence to support that CORL is more accurate than both EIASC and the Wu-Mendel Approximation, and also that there exists evidence to support the Nie-Tan Method being more accurate than both EIASC and the Wu-Mendel Approximation. The relative accuracies of the methods are summarised in Figure 6.

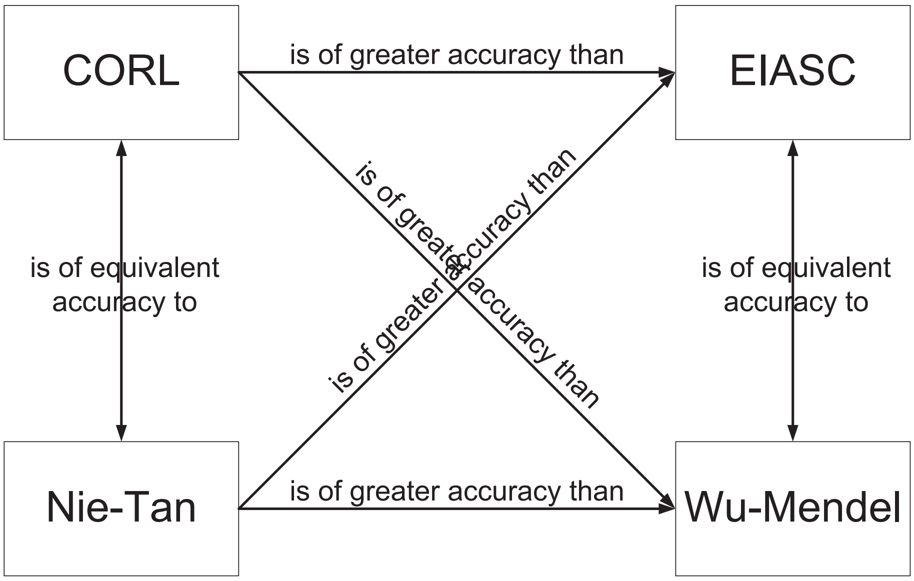


Figure 6: Wilcoxon results.

6. Efficiency of the Methods

In order to assess the efficiency of the four methods, computational complexity analyses were carried out (Tables 9 to 14). The Nie-Tan Method (Table 11), the Wu-Mendel Approximation (Table 12), and CORL (Table 13) consist of direct mappings from inputs to outputs. In contrast, EIASC is a search algorithm where the flow of control involves looping subject to testing (Subsection 3.1, Algorithm 2). The number of loops required is dependent on the set being defuzzified, so two computational complexity analyses were carried out, corresponding to the minimum possible number of loops (Table 9) and the maximum possible number of loops (Table 10). The analyses for the four methods are summarised in Table 14.

It is clear from Table 14 that the Nie-Tan Method is the simplest computationally, and the Wu-Mendel Approximation the most complex, having almost 7 times the complexity of the Nie-Tan Method. In between these two extremes, EIASC and CORL are of moderate computational complexity. EIASC is sometimes more and sometimes less complex than CORL, depending on the number of loops required for the execution of the EIASC algorithm in the interval set being defuzzified.

7. Conclusions

From this investigation we conclude that:

- The Greenfield-Chiclana Collapsing Defuzzifier (in the form of CORL) has been shown experimentally in Subsection 4.1 to be the most accurate defuzzification method of those compared. However, no statistical evidence was found to support this being the case when compared against the Nie-Tan Method.

- The ranking of the methods' computational complexity (from least computationally complex to most computationally complex) is: 1. the Nie-Tan Method, 2. the Karnik-Mendel

Stage of EIASC	$+$	$-$	$\times$	$\div$	min	max	$=$	Grand Total
Algorithm 2, Line 5	$N-1$	0	N	0	0	0		
Algorithm 2, Line 6	$N-1$	0	0	0	0	0		
Algorithm 2, Line 7	0	0	0	0	0	0	1	
Algorithm 2, Line 8	0	0	0	0	0	0	1	
Algorithm 2, Line 9 (1 loop)	1	0	0	0	0	0		
Algorithm 2, Line 10 (1 loop)	1	1	1	0	0	0		
Algorithm 2, Line 11 (1 loop)	1	1	0	0	0	0		
Algorithm 2, Line 12 (1 loop)	0	0	0	1	0	0		
Alg. 2, Line 14/18 (1 loop)	0	0	0	0	0	0	1	
Algorithm 2, Line 22	$N-1$	0	N	0	0	0		
Algorithm 2, Line 23	$N-1$	0	0	0	0	0		
Algorithm 2, Line 24	0	0	0	0	0	0	1	
Algorithm 2, Line 25	0	0	0	0	0	0	1	
Algorithm 2, Line 26 (1 loop)	1	1	1	0	0	0		
Algorithm 2, Line 27 (1 loop)	1	1	0	0	0	0		
Algorithm 2, Line 28 (1 loop)	0	0	0	1	0	0		
Algorithm 2, Line 29 (1 loop)	1	0	0	0	0	0		
Alg. 2, Line 31/35 (1 loop)	0	0	0	0	0	0	1	
Mean of l and r	1	0	0	1	0	0	0	
Totals:	$4(N-1)+7$	4	$2N+2$	3	0	0	6	$6N+14$

Table 9: Breakdown of minimum number of operations required for EIASC.

Iterative Procedure (lowest complexity possible), 3. the Greenfield-Chiclana Collapsing Defuzzifier, 4. the Karnik-Mendel Iterative Procedure (highest complexity possible), and 5. the Wu-Mendel Approximation.

- It has been demonstrated mathematically (Subsection 5.2) that in the continuous case the RESA and NTS are identical.

7.1. Recommended Interval Method

Taking both the test results and the computational complexity analyses into consideration, it is clear that the Wu-Mendel Approximation has nothing to commend it; not only does it rank as the least accurate technique, it is also the most computationally complex.

In relation to accuracy, we did not find statistical evidence to support CORL performing better than the Nie-Tan Method. This is in agreement with both methods being equivalent in the continuous case. Regarding the Wu-Mendel Approximation and EIASC, no statistical evidence was found to support one method performing better than the other. This might be explained by the Wu-Mendel Approximation being an approximation to the Karnik-Mendel Iterative Procedure. As regards the rankings of the methods, CORL was not the worst performing method in any of the comparisons, and in overall performance outranked the Nie-Tan Method. Arguably CORL is the most accurate of the four techniques.

7.2. Further Work

Out of the research presented in this paper, certain issues have emerged that would benefit from further work.

Stage of EIASC	$+$	$-$	$\times$	$\div$	min	max	$=$	Grand Total
Algorithm 2, Line 5	$N-1$	0	N	0	0	0		
Algorithm 2, Line 6	$N-1$	0	0	0	0	0		
Algorithm 2, Line 7	0	0	0	0	0	0	1	
Algorithm 2, Line 8	0	0	0	0	0	0	1	
Algorithm 2, Line 9 ($N-1$ loops)	$N-1$	0	0	0	0	0		
Algorithm 2, Line 10 ($N-1$ loops)	$N-1$	$N-1$	$N-1$	0	0	0		
Algorithm 2, Line 11 ($N-1$ loops)	$N-1$	$N-1$	0	0	0	0		
Algorithm 2, Line 12 ($N-1$ loops)	0	0	0	$N-1$	0	0		
Alg. 2, Line 14/18 ($N-1$ loops)	0	0	0	0	0	0	$N-1$	
Algorithm 2, Line 22	$N-1$	0	N	0	0	0		
Algorithm 2, Line 23	$N-1$	0	0	0	0	0		
Algorithm 2, Line 24	0	0	0	0	0	0	1	
Algorithm 2, Line 25	0	0	0	0	0	0	1	
Algorithm 2, Line 26 ($N-1$ loops)	$N-1$	$N-1$	$N-1$	0	0	0		
Algorithm 2, Line 27 ($N-1$ loops)	$N-1$	$N-1$	0	0	0	0		
Algorithm 2, Line 28 ($N-1$ loops)	0	0	0	$N-1$	0	0		
Algorithm 2, Line 29 ($N-1$ loops)	$N-1$	0	0	0	0	0		
Alg. 2, Line 31/35 ($N-1$ loops)	0	0	0	0	0	0	$N-1$	
Mean of l and r	1	0	0	1	0	0	0	
Totals:	$10(N-1)+1$	$4(N-1)$	$4N-2$	$2(N-1)+1$	0	0	$2N+2$	$22N-14$

Table 10: Breakdown of maximum number of operations required for EIASC.

Stage of Nie-Tan Method	$+$	$-$	$\times$	$\div$	min	max	Grand Total
Creation of type-1 set	N	0	0	N	0	0	
Defuzzification of type-1 set	$2(N-1)$	0	N	1	0	0	
Totals:	$3N-2$	0	N	$N+1$	0	0	$5N-1$

Table 11: Breakdown of operations required for the Nie-Tan Method.

Stage of Wu-Mendel Method	$+$	$-$	$\times$	$\div$	min	max	Grand Total
Algorithm 3, Line 10	$2(N-1)$	0	N	1	0	0	
Algorithm 3, Line 11	$2(N-1)$	0	N	1	0	0	
Algorithm 3, Line 12	0	0	0	0	1	0	
Algorithm 3, Line 13	0	0	0	0	0	1	
Algorithm 3, Line 14	$7(N-1)+1$	$3N+1$	$4N+3$	2	0	0	
Algorithm 3, Line 15	$7(N-1)+2$	$3N$	$4N+3$	2	0	0	
Algorithm 3, Line 16	1	0	0	1	0	0	
Algorithm 3, Line 17	1	0	0	1	0	0	
Mean of E_L and E_R	1	0	0	1	0	0	
Totals:	$18(N-1)+6$	$6N+1$	$10N+6$	9	1	1	$2(17N+3)$

Table 12: Breakdown of operations required for the Wu-Mendel Approximation.

Stage of Collapsing Defuzzifier	$+$	$-$	$\times$	$\div$	min	max	Grand Total
$\forall i$ calculation of b_i	0	N	0	0	0	0	
Calculation of $\|\|L\|\|$	$N-1$	0	0	0	0	0	
$\forall i$ calculation of $\|\|L\|\|+\sum_{j=1}^{i-1} r_j$	$N-1$	0	0	0	0	0	
Calculation of numerator	0	0	$N-1$	0	0	0	
Calculation of denominator	$N-1$	0	0	0	0	0	
$\forall i$ calculate r_i	0	0	0	$N-1$	0	0	
Creation of RESA	$N-1$	0	0	0	0	0	
Defuzzification of RESA	$2(N-1)$	0	N	1	0	0	
Totals For Single Pass GCCD:	$6(N-1)$	0	$2N-1$	N	0	0	$9N-7$
Totals For Two Pass CORL:	$12(N-1)$	0	$2(2N-1)$	$2N$	0	0	$18N-14$

Table 13: Breakdown of operations required for CORL.

Method	No. of Operations Required	Ranking in Order of Increasing Complexity
EIASC (fastest)	$6N+14$	2
EIASC (slowest)	$22N-14$	4
Wu-Mendel Approx.	$34N+6$	5
CORL	$18N-14$	3
Nie-Tan Method	$5N-1$	1

Table 14: Analysis of computational complexity.

Continuous Type-2 Fuzzy Inferencing. [12] and [11] summarise strong experimental evidence suggesting that the RESA's and the Nie-Tan Set's defuzzified values both approach the exhaustive defuzzified value as discretisation becomes finer. Since it has been demonstrated mathematically (Subsection 5.2) that in the continuous case the RESA and NTS are identical, were it to be proved that the Nie-Tan Set's defuzzified value approaches the exhaustive defuzzified value as discretisation becomes finer, then it would follow immediately that the continuous RESA is the RES. Therefore a mathematical proof that the continuous NTS and the continuous TRS have the same defuzzified value would be desirable. Such a proof would justify and motivate the investigation of continuous type-2 fuzzy inferencing.

Uneven performance of CORL. In Subsection 4.1, CORL is shown to be the superior method in relation to accuracy, but the technique's performance was not strong for every test set. It was observed that the sets that CORL performed well with are smooth, whereas the ones for which it performed badly are spiky. Further investigation is needed to ascertain whether the spikiness of the set defuzzified is a factor that affects the accuracy of CORL's performance, and if so, why.

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Appendix A. Interval Test Sets Figures

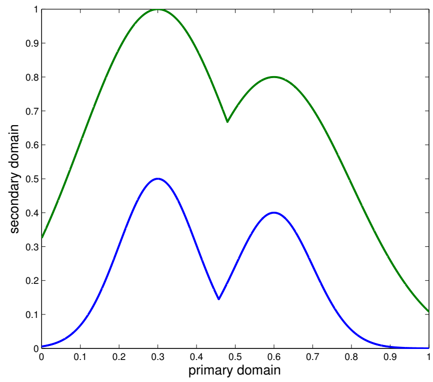

Figure A.7: Interval Test Set M.

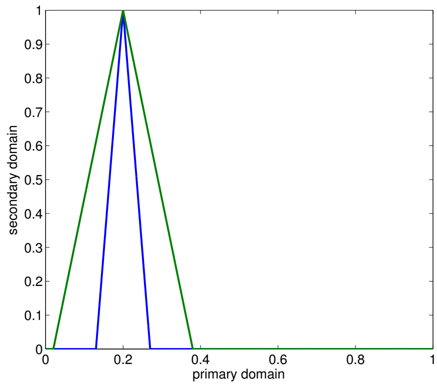

Figure A.8: Interval Test Set N.

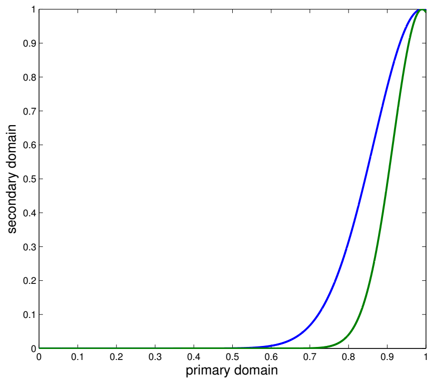

Figure A.9: Interval Test Set S.

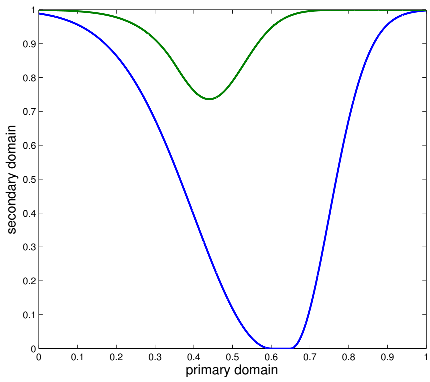

Figure A.10: Interval Test Set U.

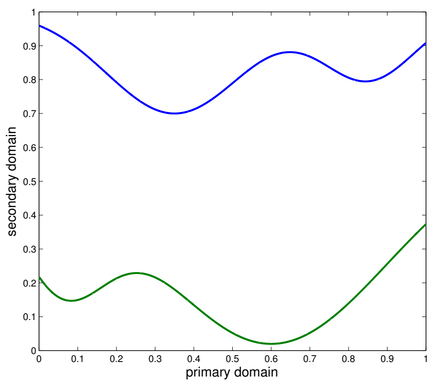

Figure A.11: Interval Test Set W.

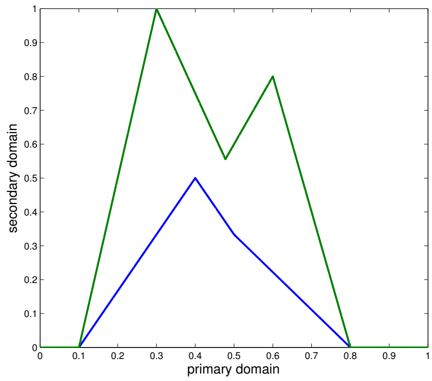

Figure A.12: Interval Test Set X.

Appendix B. Interval Test Set M

No. of Slices	Exhaustive Defuzz.	CORL	EIASC	Nie-Tan Method	Wu-Mendel Approx.
5	0.4199972460	0.4184943624	0.4211829214	0.4206867325	0.4295236614
9	0.4348736430	0.4344474812	0.4370057000	0.4337693820	0.4424349889
11	0.4358519320	0.4355179126	0.4384915906	0.4349819049	0.4433509906
17	0.4372019939	0.4369963064	0.4393687298	0.4366808102	0.4446021161
21	0.4376286265	0.4374646165	0.4397562165	0.4372193093	0.4449897526

Table B.15: Defuzzified values for test set M.

No. of Slices	Exhaustive Defuzz.	Error CORL	Error EIASC	Error Nie-Tan Method	Error Wu-Mendel App.
5	0.4199972460	0.0015028836	0.0011856754	**0.0006894865**	0.0095264154
9	0.4348736430	**0.0004261618**	0.0021320570	0.0011042610	0.0075613459
11	0.4358519320	**0.0003340194**	0.0026396586	0.0008700271	0.0074990586
17	0.4372019939	**0.0002056875**	0.0021667359	0.0005211837	0.0074001222
21	0.4376286265	**0.0001640100**	0.0021275900	0.0004093172	0.0073611261

Table B.16: Errors for test set M. The lowest errors are shown in bold.

Appendix C. Interval Test Set N

No. of Slices	Exhaustive Defuzz.	CORL	EIASC	Nie-Tan Method	Wu-Mendel Approx.
5	0.2500000000	0.2500000000	0.2500000000	0.2500000000	0.2684478216
9	0.2137762063	0.2166152302	0.2135859199	0.2071078431	0.2592275919
11	0.2000000000	0.1996177735	0.2000000000	0.2000000000	0.2094094585
17	0.2003511054	0.1997796246	0.1999873891	0.1998318386	0.2205701965
21	0.2000000000	0.1996245463	0.2000000000	0.2000000000	0.2162988115

Table C.17: Defuzzified values for test set N.

No. of Slices	Exhaustive Defuzz.	Error CORL	Error EIASC	Error Nie-Tan Method	Error Wu-Mendel App.
5	0.2500000000	**0.0000000000**	**0.0000000000**	**0.0000000000**	0.0184478216
9	0.2137762063	0.0028390239	**0.0001902864**	0.0066683632	0.0454513856
11	0.2000000000	0.0003822265	**0.0000000000**	**0.0000000000**	0.0094094585
17	0.2003511054	0.0005714808	**0.0003637163**	0.0005192668	0.0202190911
21	0.2000000000	0.0003754537	**0.0000000000**	**0.0000000000**	0.0162988115

Table C.18: Errors for test set N. The lowest errors are shown in bold.

Appendix D. Interval Test Set S

No. of Slices	Exhaustive Defuzz.	CORL	EIASC	Nie-Tan Method	Wu-Mendel Approx.
5	0.9819755268	0.9819724508	0.9820266266	0.9808139836	0.9807620182
9	0.9480255757	0.9479604407	0.9498630718	0.9466403800	0.9457130018
11	0.9411164375	0.9410330760	0.9433286138	0.9399863542	0.9388383001
17	0.9309357359	0.9308645243	0.9143284643	0.9301283898	0.9285457815
21	0.9273868277	0.9273228598	0.9109648792	0.9267196454	0.9249480230

Table D.19: Defuzzified values for test set S.

No. of Slices	Exhaustive Defuzz.	Error CORL	Error EIASC	Error Nie-Tan Method	Error Wu-Mendel App.
5	0.9819755268	**0.0000030760**	0.0000510998	0.0011615432	0.0012135086
9	0.9480255757	**0.0000651350**	0.0018374961	0.0013851957	0.0023125739
11	0.9411164375	**0.0000833615**	0.0022121763	0.0011300833	0.0022781374
17	0.9309357359	**0.0000712116**	0.0166072716	0.0008073461	0.0023899544
21	0.9273868277	**0.0000639679**	0.0164219485	0.0006671823	0.0024388047

Table D.20: Errors for test set S. The lowest errors are shown in bold.

Appendix E. Interval Test Set U

No. of Slices	Exhaustive Defuzz.	CORL	EIASC	Nie-Tan Method	Wu-Mendel Approx.
5	0.4883524681	0.4883390379	0.4872834960	0.4898381531	0.4886512972
9	0.4901791343	0.4901282562	0.4882247910	0.4912375749	0.4898251596
11	0.4897646372	0.4897114396	0.4874545055	0.4907033605	0.4891705815
17	0.4896410219	0.4895948384	0.4868477729	0.4902778680	0.4886038758
21	0.4895764854	0.4895352299	0.4866154107	0.4901020305	0.4883730907

Table E.21: Defuzzified values for test set U.

No. of Slices	Exhaustive Defuzz.	Error CORL	Error EIASC	Error Nie-Tan Method	Error Wu-Mendel App.
5	0.4883524681	**0.0000134302**	0.0010689721	0.0014856850	0.0002988291
9	0.4901791343	**0.0000508781**	0.0019543433	0.0010584406	0.0003539747
11	0.4897646372	**0.0000531976**	0.0023101317	0.0009387233	0.0005940557
17	0.4896410219	**0.0000461835**	0.0027932490	0.0006368461	0.0010371461
21	0.4895764854	**0.0000412555**	0.0029610747	0.0005255451	0.0012033947

Table E.22: Errors for test set U. The lowest errors are shown in bold.

Appendix F. Interval Test Set W

No. of Slices	Exhaustive Defuzz.	CORL	EIASC	Nie-Tan Method	Wu-Mendel Approx.
5	0.5113179745	0.5113040394	0.5161402998	0.5077561025	0.5105303188
9	0.5061788619	0.5063274201	0.5073309871	0.5049025262	0.5069806367
11	0.5053236036	0.5054813865	0.5058808491	0.5044228942	0.5062420655
17	0.5040847945	0.5042208959	0.5038132389	0.5036577343	0.5050172344
21	0.5036907192	0.5038093313	0.5031529409	0.5033898118	0.5045764132

Table F.23: Defuzzified values for test set W.

No. of Slices	Exhaustive Defuzz.	Error CORL	Error EIASC	Error Nie-Tan Method	Error Wu-Mendel App.
5	0.5113179745	**0.0000139351**	0.0048223253	0.0035618720	0.0007876557
9	0.5061788619	**0.0001485582**	0.0011521252	0.0012763357	0.0008017748
11	0.5053236036	**0.0001577829**	0.0005572455	0.0009007094	0.0009184619
17	0.5040847945	**0.0001361014**	0.0002715556	0.0004270602	0.0009324399
21	0.5036907192	**0.0001186121**	0.0005377783	0.0003009074	0.0008856940

Table F.24: Errors for test set W. The lowest errors are shown in bold.

Appendix G. Interval Test Set X

No. of Slices	Exhaustive Defuzz.	CORL	EIASC	Nie-Tan Method	Wu-Mendel Approx.
5	0.4213200635	0.4210574784	0.4178281069	0.4149746193	0.4232352457
9	0.4343761342	0.4338356112	0.4344020217	0.4346006144	0.4391219149
11	0.4323638373	0.4318365393	0.4321428571	0.4322643343	0.4373876062
17	0.4312767130	0.4309748555	0.4317442643	0.4312318453	0.4364261865
21	0.4322207919	0.4319600819	0.4325327375	0.4321864324	0.4372554986

Table G.25: Defuzzified values for test set X.

No. of Slices	Exhaustive Defuzz.	Error CORL	Error EIASC	Error Nie-Tan Method	Error Wu-Mendel App.
5	0.4213200635	**0.0002625851**	0.0034919566	0.0063454442	0.0019151822
9	0.4343761342	0.0005405230	**0.0000258875**	0.0002244802	0.0047457807
11	0.4323638373	0.0005272980	0.0002209802	**0.0000995030**	0.0050237689
17	0.4312767130	0.0003018575	0.0004675513	**0.0000448677**	0.0051494735
21	0.4322207919	0.0002607100	0.0003119456	**0.0000343595**	0.0050347067

Table G.26: Errors for test set X. The lowest errors are shown in bold.